The Exponential Shift Theorem

There is a particularly useful theorem, called the *Exponential Shift Theorem* that results from the Product Rule that you learned about in first year calculus.

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x) + g(x)f'(x)$$

Let's use the notation D instead of $\frac{d}{dx}$.
Also, take the special case where $g(x) = e^{rx}$ (r is a constant).

$$D\left(e^{rx}f(x)\right) = re^{rx}f(x) + e^{rx}f'(x)$$

If we rewrite this relationship using operator notation, we get:

$$D\left(e^{rx}f(x)\right) = e^{rx}(D+r)f(x) \qquad \mathbf{(1)}$$

Equation (**1**) is a special case of the formula that we will call the Exponential Shift Theorem. To generalize equation (**1**), consider what happens if we replace the operator D with the operator D^2.

$$D^2\left(e^{rx}f(x)\right) = D(D\left(e^{rx}f(x)\right) = D\left(e^{rx}(D+r)f(x)\right)$$

The last expression on the right of this equation comes from equation **(1)**. Now, apply equation **(1)** again with $(D+r)f(x)$ instead of $f(x)$

$$D^2\left(e^{rx}f(x)\right) = e^{rx}(D+r)(D+r)f(x) = e^{rx}(D+r)^2f(x)$$

We can repeat this calculation in the same way with the operator D^3

$$D^3\left(e^{rx}f(x)\right) = D(D^2\left(e^{rx}f(x)\right)) = D\left(e^{rx}(D+r)^2f(x)\right) = e^{rx}(D+r)^3f(x)$$

More generally,

$$D^k\left(e^{rx}f(x)\right) = e^{rx}(D+r)^kf(x) \qquad \mathbf{(2)}$$

Example 1.
If $y = x^4e^x$, calculate the third derivative.

Solution:

$$D^3y = D^3\left(x^4e^x\right) = e^x(D+1)^3x^4 = e^x(D^3+3D^2+3D+1)(x^4) = e^x(24x+36x^2+12x^3+x^4)$$

We can generalize equation **(2)** even further by recognizing that *any* linear differential operator is a combination of terms of the form D^k.

Let $P(t)$ be the following polynomial:

$$P(t) = a_nt^n + a_{n-1}t^{n-1} + \cdots a_1t + a_0 = \sum_{k=0}^{n} a_kt^k$$

If we replace each occurence of t in this polynomial with the operator D we obtain a differential operator $P(D)$

$$P(D) = a_n D^n + a_{n-1} D^{n-1} + \cdots a_1 D + a_0 = \sum_{k=0}^{n} a_k D^k$$

Now, apply this operator to an expression of the form $e^{rx} f(x)$

$$\begin{aligned} P(D)\left(e^{rx} f(x)\right) &= \sum_{k=0}^{n} a_k D^k \left(e^{rx} f(x)\right) \\ &= \sum_{k=0}^{n} a_k e^{rx} (D+r)^k f(x) \qquad \text{(This follows from equation } \mathbf{(2)}\text{)} \\ &= e^{rx} \sum_{k=0}^{n} a_k (D+r)^k f(x) \\ &= e^{rx} P(D+r) f(x) \end{aligned}$$

We have just discovered the following formula:

$$P(D)\left(e^{rx} f(x)\right) = e^{rx} P(D+r) f(x) \tag{3}$$

Equation **(3)** is the *Exponential Shift Theorem.*

Example 2.
Let $y = e^{-x} \sin x$. Calculate the expression $y'' + y'$

Solution:

$$\begin{aligned} (D^2 + D)y &= (D^2 + D)\left(e^{-x} \sin x\right) \\ &= e^{-x}\left((D-1)^2 + (D-1)\right)(\sin x) \\ &= e^{-x}\left(D^2 - D\right)(\sin x) \\ &= e^{-x}(-\sin x - \cos x) \end{aligned}$$

Example 3.
Let $y = x \cosh x$. Calculate the expression $\frac{d^4 y}{dx^4} - y$

Solution:

$$\begin{aligned} (D^4 - 1)\left(x \cdot \frac{1}{2}\left(e^x + e^{-x}\right)\right) &= \frac{1}{2}(D^4 - 1)(xe^x) + \frac{1}{2}(D^4 - 1)(xe^{-x}) \\ &= \frac{1}{2} e^x ((D+1)^4 - 1)(x) + \frac{1}{2}((D-1)^4 - 1)(x) \\ &= \frac{1}{2} e^x (D^4 + 4D^3 + 6D^2 + 4D)(x) + \frac{1}{2} e^{-x} (D^4 - 4D^3 + 6D^2 - 4D)(x) \\ &= \frac{1}{2} e^x (4) + \frac{1}{2} e^{-x} (-4) \\ &= 2e^x - 2e^{-x} = 4 \sinh x \end{aligned}$$

Example 4.
Solve the differential equation:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

If we substitute e^{rx} into this equation, we obtain:

$$r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0$$

$$r = -1$$

Thus, e^{-x} is a solution. However, to find the general solution of this second order equation, we need another solution independent of the first one. There is a clever substitution that, when combined with the Exponential Shift Theorem, produces all the solutions of the differential equation.

Let $u = e^x y$. This permits us to substitute $e^{-x}u$ in place of y in the differential equation.

$$(D+1)^2 y = 0$$

$$(D+1)^2 \left(e^{-x}u\right) = 0$$

$$e^{-x} D^2 u = 0$$

$$D^2 u = 0$$

$$Du = C_1$$

$$u = C_1 x + C_2$$

$$y = e^{-x}u = C_1 x e^{-x} + C_2 e^{-x}$$

We have obtained e^{-x}, which we already knew about. However, we have also obtained xe^{-x}, which we did not know about at all.

Example 5
Solve the differential equation:

$$(D-4)^3 y = 0$$

We can see that e^{4x} is going to be a solution, but what are the other solutions? Let $u = e^{-4x}y$ and substitute into the equation.

$$(D-4)^3 \left(e^{4x}u\right) = 0$$

$$e^{4x} D^3 u = 0$$

$$D^3 u = 0$$

Now, integrate both sides three times to obtain:

$$u = a + bx + cx^2$$

It follows that the general solution of the differential equation is:

$$y = e^{4x}u = ae^{4x} + bxe^{4x} + cx^2 e^{4x}$$